STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

ZW 1957-012

Supplement to Rapport TW 33

Asymptotic expansion of a certain Fourier Coefficient

C.G. Lekkerkerker and A.H.M. Levelt

1957

Supplement to Rapport TW 33. Asymptotic expansion of a certain Fourier coefficient

by

C.G. Lekkerkerker and A.H.M. Levelt

In his report "The expansion of a function into a Fourier series with prescribed phases valid in the half-period interval" (Rapport TW 33), Dr H.A. Lauwerier considered among other things the problem of expanding a function $f(x)$ in the interval $(0,\pi)$ into a series of the form:

$$(1)\qquad f(x) = \sum_{n=1}^{\infty} b_n \sin(nx + \mu\pi),$$

where μ is a real or complex constant with $\mu \neq 0$, $|\mathrm{Re}\,\mu| < \frac{1}{2}$. To this end he introduced the function

$$\phi_o(z) = \sum_{n=1}^{\infty} b_n e^{inz} \qquad (\mathrm{Im}\, z > 0).$$

Defining $\phi(w)$ and $\varphi(t)$ by

$$\phi(-\cos z) = \phi_o(z)$$

$$\varphi(-\cos x) = f(x)\ ,$$

he proved that

$$\phi(w) = \left(\frac{w+1}{w-1}\right)^{\mu} \frac{1}{\pi} \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)}{t-w}\, dt\ .$$

Thus the coefficients b_n in (1) are given by

$$(2)\qquad b_n = \frac{1}{2\pi i} \oint \phi\left(-\tfrac{1}{2}\left(s+\tfrac{1}{s}\right)\right) s^{-n-1}\, ds$$

$$= \frac{1}{2\pi i} \oint \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \frac{1}{\pi} \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)}{t+\frac{1}{2}\left(s+\frac{1}{s}\right)}\, dt$$

(see formulas 3-15 and 5-2 in the named report). These formulas hold under rather general conditions for the function $\varphi(t)$, e.g. that it be differentiable in $[0,\pi]$.

Lauwerier further investigated the behaviour of b_n for large n. He stated without proof that e.g. if $f(x)$ is differentiable at $x=\pi$, one has [1])

---

1) Lauwerier has as O-term $O(n^{-3\pm 2\mu_0})$, which certainly is incorrect. For the last line of p.13 should read

$$\left(\frac{e^u+1}{e^u-1}\right)^{\alpha} = \left(\frac{2}{u}\right)^{\alpha} \left\{1+O(u)\right\}\ .$$

$$(3)\qquad b_n = \frac{(-1)^{n+1}\, 2^{2\mu}}{\Gamma(2\mu)}\, \frac{A}{n^{1-2\mu}} + \frac{2^{-2\mu}}{\Gamma(-2\mu)}\, \frac{B}{n^{1+2\mu}} + O(n^{-2+2|\mu_0|}),$$

where

$$(4)\qquad A = -\frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)-\varphi(1)}{t-1}\,dt + \frac{\varphi(1)}{\sin\mu\pi}$$

$$(5)\qquad B = \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)-\varphi(-1)}{t+1}\,dt - \frac{\varphi(-1)}{\sin\mu\pi}$$

(see formulas 5-8 and 6-3). In this report a rigorous proof of (3) will be given and even an asymptotic expansion of b_n will be obtained (see (17)).

In the case that $\varphi(t)$ is analytic one can modify the path of integration in the inner integral and thus obtain the analytic continuation of this integral as a function of s over the whole s-plane with the exception of the points $s=\pm1$. Then one can apply the method explained in a special case in section 4 of the report mentioned, dealing with integrals of the form $\frac{1}{2\pi i}\oint \frac{1}{s^{n+1}}\,\psi(s)ds$, where $\psi(s)$ is analytic for $s\neq \pm1$. In our case, where we do not assume analyticity of $\varphi(t)$, we have to follow another procedure.

Throughout this report we shall suppose that $\varphi(t)$ is integrable over $(-1,1)$ and that at $t=-1$ the first **k** right-hand derivatives of $\varphi(t)$ and at $t=1$ the first k left-hand derivatives of $\varphi(t)$ exist, where k is some positive integer. We then can find a sequence of polynomials $g_1(t), g_2(t), \ldots, g_{2k}(t)$ satisfying the following requirements:

1^0. $g_j(t)$ is a polynomial of degree $j-1$ $(j=1,2,\ldots,2k)$

2^0. at the endpoints one has, for $i=1,2,\ldots,k$,

$$\varphi(t)-g_{2i-1}(t) = \begin{cases} O((1+t)^{i-1}) & \text{as } t\to -1 \\ O((1-t)^{i}) & \text{as } t\to 1 \end{cases}$$

$$\varphi(t)-g_{2i}(t) = \begin{cases} O((1+t)^{i}) & \text{as } t\to -1 \\ O((1-t)^{i}) & \text{as } t\to 1\,. \end{cases}$$

In particular, $g_1(t)=\varphi(-1)$. Further we shall denote by c_j the highest coefficient in $g_j(t)$ $(j=1,2,\ldots,2k)$. We finally put $\mathrm{Re}\,\mu = \mu_0$.

With the above assumptions and notations we shall derive the following asymptotic expansion:

$$(6)\qquad b_n = \frac{2}{\pi}\int_0^{\pi}\left\{\varphi(-\cos x)-g_{2k}(-\cos x)\right\}\sin(nx+\mu\pi)dx + \\ + \sum_{p=1}^{k}\left\{a_p(n)\,A_p + b_p(n)B_p\right\} + O\left(\frac{1}{n^{k-\frac{1}{2}}}\right),$$

where

$$(7)\quad a_p(n) = \frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \frac{(s+1)^{2p-2}(s-1)^{2p-2}}{(2s)^{2p-2}} ,$$

$$(8)\quad b_p(n) = -\frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \frac{(s+1)^{2p}(s-1)^{2p-2}}{(2s)^{2p-1}} ,$$

$$(9)\quad A_p = \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)-g_{2p-1}(t)}{(t-1)^p(t+1)^{p-1}}\, dt - \frac{c_{2p-1}}{\sin\mu\pi} ,$$

$$(10)\quad B_p = \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)-g_{2p}(t)}{(t-1)^p(t+1)^{p}}\, dt - \frac{c_{2p}}{\sin\mu\pi} .$$

We shall first deduce (6) and then discuss this result. For the proof of (6) we shall use that

$$(11)\qquad a_p(n) = O\left(\frac{1}{n^{2p-1-2\,|\mu_0|}}\right), \quad b_p(n) = O\left(\frac{1}{n^{2p-1+2\mu_0}}\right) .$$

These estimates are obtained as explained in Rapport TW 33 (in particular § 4), and will also be deduced at the end of this report.

Further we shall use a certain mixed expansion of $\frac{1}{t-z}$. We have

$$(12)\qquad \frac{1}{t-z} = \frac{1}{t-1} + \frac{z-1}{(t-1)(t-z)}$$

and also

$$(12')\qquad \frac{1}{t-z} = \frac{1}{t+1} + \frac{z+1}{(t+1)(t-z)} .$$

Applying alternately (12) and (12') we get

$$(13)\qquad \frac{1}{t-z} = \sum_{p=1}^{k} \frac{(z-1)^{p-1}(z+1)^{p-1}}{(t-1)^p(t+1)^{p-1}} + \sum_{p=1}^{k} \frac{(z-1)^{p}(z+1)^{p-1}}{(t-1)^p(t+1)^{p}} + \frac{(z-1)^k(z+1)^k}{(t-1)^k(t+1)^k}\cdot\frac{1}{t-z} .$$

We shall successively deal with the integral in the right-hand member of (2) with $\varphi(t)$ replaced by $g_{2k}(t)$ and $\varphi(t)-g_{2k}(t)$ respectively.

I. Put $b'_n = \frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \cdot \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} g_{2k}(t)\, \frac{dt}{t+\frac{1}{2}(s+\frac{1}{s})}$.

Further write

$$I(z) = \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} g_{2k}(t)\, \frac{dt}{t-z} \qquad (z \notin [-1,1]),$$

$$F(z) = \frac{1}{2\pi i}\int_{C} \left(\frac{w-1}{w+1}\right)^{\mu} g_{2k}(w)\, \frac{dw}{w-z} ,$$

where C is a simple contour around the points -1 and 1, which does not enclose the point z. We have

$$F(z) = \frac{1}{2i}\left(e^{-\mu\pi i}-e^{\mu\pi i}\right)I(z) = -\sin\mu\pi \,.\, I(z).$$

Further, using (13), with t replaced by w, we have

$$F(z) = \sum_{j=1}^{2k} F_j(z) + R(z),$$

where

$$F_{2p-1}(z) = \frac{1}{2\pi i}\int_C \left(\frac{w-1}{w+1}\right)^{\mu} g_{2k}(w)\, \frac{(z-1)^{p-1}(z+1)^{p-1}}{(w-1)^p(w+1)^{p-1}}\, dw,$$

$$F_{2p}(z) = \frac{1}{2\pi i}\int_C \left(\frac{w-1}{w+1}\right)^{\mu} g_{2k}(w)\, \frac{(z-1)^{p}(z+1)^{p-1}}{(w-1)^p(w+1)^{p}}\, dw,$$

$$R(z) = \frac{1}{2\pi i}\int_C \left(\frac{w-1}{w+1}\right)^{\mu} g_{2k}(w)\, \frac{(z-1)^{k}(z+1)^{k}}{(w-1)^k(w+1)^{k}}\, \frac{dw}{w-z} .$$

Now, by the calculus of residues,

$$\frac{1}{2\pi i}\int_C \left(\frac{w-1}{w+1}\right)^{\mu} g_{2p-1}(w)\, \frac{dw}{(w-1)^p(w+1)^{p-1}} = c_{2p-1},$$

$$\frac{1}{2\pi i}\int_C \left(\frac{w-1}{w+1}\right)^{\mu} g_{2p}(w)\, \frac{dw}{(w-1)^p(w+1)^{p}} = c_{2p} ,$$

$$\frac{1}{2\pi i}\int_C \left(\frac{w-1}{w+1}\right)^{\mu} g_{2k}(w)\, \frac{dw}{(w-1)^k(w+1)^{k}(w-z)} = -\left(\frac{z-1}{z+1}\right)^{\mu} \frac{g_{2k}(z)}{(z-1)^k(z+1)^k} ,$$

since the residues at the point $w=\infty$ are successively equal to c_{2p-1}, c_{2p}, 0. Hence, by 2°,

$$F_{2p-1}(z) = (z-1)^{p-1}(z+1)^{p-1}\left\{c_{2p-1} - \frac{\sin\mu\pi}{\pi}\int_1^1 \left(\frac{1-t}{1+t}\right)^{\mu} \frac{g_{2k}(t)-g_{2p-1}(t)}{(t-1)^p(t+1)^{p-1}}\, dt\right\}$$

$$F_{2p}(z) = (z-1)^{p}(z+1)^{p-1}\left\{c_{2p} - \frac{\sin\mu\pi}{\pi}\int_{-1}^1 \left(\frac{1-t}{1+t}\right)^{\mu} \frac{g_{2k}(t)-g_{2p}(t)}{(t-1)^p(t+1)^{p}}\, dt\right\},$$

$$R(z) = -\left(\frac{z-1}{z+1}\right)^{\mu} g_{2k}(z).$$

So we find

(14) $$b_n' = \frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}}\left(\frac{1-s}{1+s}\right)^{2\mu} . I\left(-\tfrac{1}{2}\left(s+\tfrac{1}{s}\right)\right)$$

$$= \frac{-1}{\sin\mu\pi}\,\frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}}\left(\frac{1-s}{1+s}\right)^{2\mu} \cdot$$

$$\cdot \left\{\sum_{p=1}^{k} F_{2p-1}\left(-\tfrac{1}{2}\left(s+\tfrac{1}{s}\right)\right) + \sum_{p=1}^{k} F_{2p}\left(-\tfrac{1}{2}\left(s+\tfrac{1}{s}\right)\right) + R\left(-\tfrac{1}{2}\left(s+\tfrac{1}{s}\right)\right)\right\}$$

$$= \sum_{p=1}^{k}\left\{a_p(n)A_p' + b_p(n)B_p'\right\},$$

where

$$A_p' = \frac{1}{\pi}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{\mu} \frac{g_{2k}(t)-g_{2p-1}(t)}{(t-1)^p(t+1)^{p-1}}\, dt - \frac{c_{2p-1}}{\sin\mu\pi} ,$$

$$B'_p = \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{g_{2k}(t)-g_{2p}(t)}{(t-1)^p(t+1)^p}\, dt - \frac{c_{2p}}{\sin\mu\pi},$$

and $a_p(n)$ and $b_p(n)$ are given by (7) and (8).
Here we have used that

$$\frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} R(-\tfrac{1}{2}(s+\tfrac{1}{s})) = -\frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}}\, g_{2k}(-\tfrac{1}{2}(s+\tfrac{1}{s})) = 0.$$

II Put

$$b''_n = \frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \psi(t)\, \frac{dt}{t+\frac{1}{2}(s+\frac{1}{s})},$$

where $\psi(t) = \varphi(t)-g_{2k}(t)$. Further take $\delta = \frac{1}{n}$ and write

$$b''_n = \frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \cdot \frac{1}{\pi}\left\{\int_{-1+\delta}^{1-\delta} + \int_{-1}^{-1+\delta} + \int_{1-\delta}^{1}\right\} = I_1+I_2+I_3,$$

say. We first deal with I_1. In the expression for this quantity we interchange the two integrations and then modify in a certain way to be described below the path of integration with respect to s. For fixed t the factor $\frac{1}{t+\frac{1}{2}(s+\frac{1}{s})}$ has two simple poles at the points s_1 and s_2 given by

$$s_{1,2} = -t \pm i\sqrt{1-t^2} \qquad (\operatorname{Im} s_1 > 0,\ \operatorname{Im} s_2 < 0).$$

One has

$$\frac{1}{t+\frac{1}{2}(s+\frac{1}{s})} = \frac{2s}{(s-s_1)(s-s_2)} = \frac{2s}{s-s_2}\left\{\frac{1}{s-s_1} - \frac{1}{s-s_2}\right\}.$$

Further, $t=-\frac{1}{2}(s_1+\frac{1}{s_1}) = -\frac{1}{2}(s_2+\frac{1}{s_2})$, hence

$$\frac{t+1}{t-1} = \left(\frac{1-s_1}{1+s_1}\right)^2 = \left(\frac{1-s_2}{1+s_2}\right)^2.$$

Next, $\left(\frac{1-s_1}{1+s_1}\right)^{2\mu} = e^{-\mu\pi i}\left(\frac{1+t}{1-t}\right)^{\mu}$, $\left(\frac{1-s_2}{1+s_2}\right)^{2\mu} = e^{\mu\pi i}\left(\frac{1+t}{1-t}\right)^{\mu}$.

Hence the sum of the residues of

$\frac{1}{s^{n+1}}\left(\frac{1-s}{1+s}\right)^{2\mu}\frac{1}{t+\frac{1}{2}(s+\frac{1}{s})}$ at the points s_1 and s_2 is equal to

$$2\left(\frac{1+t}{1-t}\right)^{\mu} \cdot \frac{1}{s_1-s_2}\left\{e^{-\mu\pi i}\, s_1^{-n} - e^{\mu\pi i}\, s_2^{-n}\right\}$$

$$= -2\left(\frac{1+t}{1-t}\right)^{\mu} \frac{\sin(nx+\mu\pi)}{\sin x}, \text{ if } s_1=e^{ix} \text{ (and so } s_2=e^{-ix}).$$

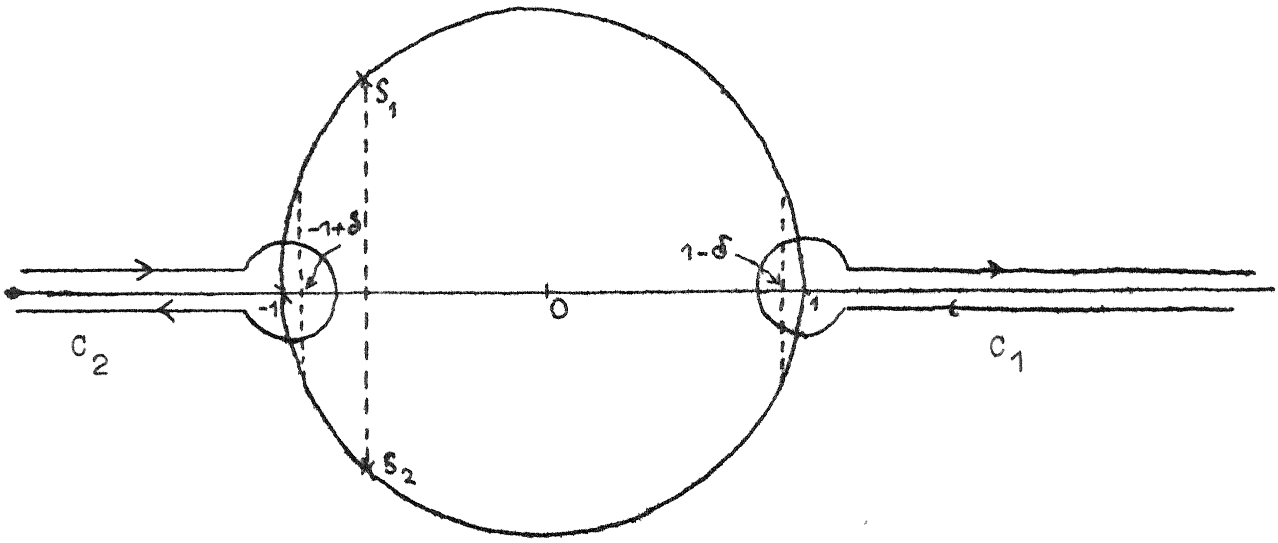
So we find

$$(15)\quad I_1 = \frac{1}{\pi}\int_{-1+\delta}^{1-\delta} \left(\frac{1-t}{1+t}\right)^{\mu} \psi(t)dt \cdot \frac{1}{2\pi i}\oint \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \frac{1}{t+\frac{1}{2}(s+\frac{1}{s})}$$

$$= \frac{2}{\pi}\int_{-1+\delta}^{1-\delta} \psi(t)\ \frac{\sin(nx+\mu\pi)}{\sin x}\ dt$$

$$+ \frac{1}{\pi}\int_{-1+\delta}^{1-\delta} \left(\frac{1-t}{1+t}\right)^{\mu} \psi(t)dt \cdot \frac{1}{2\pi i}\int_{C_1+C_2} \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \frac{1}{t+\frac{1}{2}(s+\frac{1}{s})},$$

where x is determined by $-\cos x = -\frac{1}{2}(s_1+\frac{1}{s_1})=t$ and where C_1 denotes a contour from $+\infty$ to $+\infty$, which encloses the interval $[1.\infty]$ on the real axis and which does not enclose the points $s_{1,2}$ for any t with $-1+\delta \leqslant t \leqslant 1-\delta$, and C_2 a similar contour from $-\infty$ to $-\infty$ around the point -1 (see figure 1).



We now compute

$$\frac{1}{\pi}\int_{-1+\delta}^{1-\delta} \left(\frac{1-t}{1+t}\right)^{\mu} \psi(t)dt \cdot \frac{1}{2\pi i}\int_{C_1} \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \frac{1}{t+\frac{1}{2}(s+\frac{1}{s})} = I_1',$$

say. Using the expansion (13), with $z=-\frac{1}{2}(s+\frac{1}{s})$, we get

$$I_1' = \sum_{j=1}^{2k} G_j + S,$$

where

$$G_{2p-1}= \frac{1}{\pi}\int_{-1+\delta}^{1-\delta} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{(t-1)^p(t+1)^{p-1}}\,dt \,\cdot$$

$$\cdot \frac{1}{2\pi i}\int_{C_1} \frac{1}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu}\left(-\tfrac{1}{2}(s+\tfrac{1}{s})-1\right)^{p-1}\left(-\tfrac{1}{2}(s+\tfrac{1}{s})+1\right)^{p-1}ds,$$

$$G_{2p}= \frac{1}{\pi}\int_{-1+\delta}^{1-\delta} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)dt}{(t-1)^p(t+1)^p} \,\cdot$$

$$\cdot \frac{1}{2\pi i}\int_{C_1} \frac{1}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu}\left(-\tfrac{1}{2}(s+\tfrac{1}{s})-1\right)^{p}\left(-\tfrac{1}{2}(s+\tfrac{1}{s})+1\right)^{p-1} ds,$$

$$S = \frac{1}{\pi}\int_{-1+\delta}^{1-\delta} dt\left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{(t-1)^k(t+1)^k} \; \frac{1}{2\pi i}\int_{C_1} \frac{1}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \cdot$$

$$\cdot \frac{\left(-\frac{1}{2}(x+\frac{1}{s})-1\right)^k\left(-\frac{1}{2}(s+\frac{1}{s})+1\right)^k}{t+\frac{1}{2}(s+\frac{1}{s})} ds .$$

The last expression can be estimated as follows. We have

$$-\tfrac{1}{2}(s+\tfrac{1}{s})-1 = -\left(\frac{s+1}{2s}\right)^2, \quad -\tfrac{1}{2}(s+\tfrac{1}{s})+1 = -\frac{(s-1)^2}{2s},$$

$$\left|t+\tfrac{1}{2}(s+\tfrac{1}{s})\right| \geq 1-t \text{ for } s \geq 1,$$

hence

$$\left|\frac{1}{2\pi i}\int_{C_1}\right| \leq \frac{1}{\pi}\int_1^{\infty} \frac{1}{s^{n+1}} \left|\frac{1-s}{1+s}\right|^{2\mu_0} \frac{(s+1)^{2k}(s-1)^{2k}}{(2s)^{2k}} ds \cdot \frac{1}{1-t}$$

$$= \frac{1}{\pi 2^{2k}(1-t)}\int_1^{\infty} \frac{(s-1)^{2k+2\mu_0}(s+1)^{2k-2\mu_0}}{s^{n+1+2k}} ds =$$

$$= O\left(\frac{1}{(1-t)n^{2k+1+2\mu_0}}\right) .$$

Hence, since $1-t \geq \delta = \frac{1}{n}$,

$$\frac{1}{2\pi i}\int_{C_1} = O\left(\frac{1}{n^{2k+2\mu_0}}\right),$$

and so

$$S = O\left\{\frac{1}{n^{2k+2\mu_0}}\int_{-1+\delta}^{1-\delta} \left(\frac{1-t}{1+t}\right)^{\mu_0} \frac{|\psi(t)|}{(1-t)^k(1+t)^k} dt\right\}$$

$$= O\left(\frac{1}{n^{2k+2\mu_0}}\right) = O\left(\frac{1}{n^{2k-1}}\right),$$

because $|\mu_0| = |\mathrm{Re}\,\mu| \leq \frac{1}{2}$ and $\frac{\psi(t)}{(1-t)^k(1+t)^k}$ is bounded in the interval $(-1,1)$, by the choice of $g_{2k}(t)$.

In a similar way we can treat

$$I_1'' = \frac{1}{\pi}\int_{-1+\delta}^{1-\delta} \left(\frac{1-t}{1+t}\right)^{\mu} \psi(t)dt \cdot \frac{1}{2\pi i}\int_{C_2} \frac{ds}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \frac{1}{t+\frac{1}{2}(s+\frac{1}{s})} .$$

Since

$$\frac{1}{2\pi i}\int_{C_1+C_2} \frac{1}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \left(-\tfrac{1}{2}(s+\tfrac{1}{s})-1\right)^{p-1}\left(-\tfrac{1}{2}(s+\tfrac{1}{s})+1\right)^{p-1} ds = a_p(n),$$

$$\frac{1}{2\pi i}\int_{C_1+C_2} \frac{1}{s^{n+1}} \left(\frac{1-s}{1+s}\right)^{2\mu} \left(-\tfrac{1}{2}(s+\tfrac{1}{s})-1\right)^{p}\left(\tfrac{1}{2}(s+\tfrac{1}{s})+1\right)^{p-1} ds = b_p(n),$$

we get

$$I_1' + I_1'' = \sum_{p=1}^{k} \left\{a_p(n)A_p'' + b_p(n)B_p''\right\} + O\left(\frac{1}{n^{2k-1}}\right) ,$$

where
$$A_p'' = \frac{1}{\pi}\int_{-1+\delta}^{1-\delta} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{(t-1)^p(t+1)^{p-1}}\,dt,$$

$$B_p'' = \frac{1}{\pi}\int_{-1+\delta}^{1-\delta} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{(t-1)^p(t+1)^{p}}\,dt\ .$$

Next, we estimate I_2 and I_3. For the path of integration with respect to s we take the circle around 0 with radius $1-\frac{2}{n}$. We then have $\left|t+\frac{1}{2}(s+\frac{1}{s})\right| \geqq \frac{1}{2}\left|s^2+2ts+1\right| = \frac{1}{2}\left|(s-s_1)(s-s_2)\right| \geqq \frac{1}{5}|s-1|^2$, hence

$$\int_{-1}^{-1+\delta} \left(\frac{1-t}{1+t}\right)^{\mu} \psi(t)\,\frac{dt}{t+\frac{1}{2}(s+\frac{1}{s})} = O\left(\delta^{k-\mu_0+1}\,\frac{1}{|s-1|^2}\right),$$

and so
$$I_2 = O\left\{\delta^{k-\mu_0+1}\int_0^{2\pi} \frac{\left|(1-\frac{2}{n})e^{i\theta}-1\right|^{2\mu_0-2}}{\left|(1-\frac{2}{n})e^{i\theta}+1\right|^{2\mu_0}}\,d\theta\right\}$$
$$= O\left\{\delta^{k-\mu_0+1}\,.\,\left(\frac{1}{n}\right)^{\min(2\mu_0-1,\,1-2\mu_0)}\right\}$$
$$= O\left(\frac{1}{n^{k+\mu_0}}\right) = O\left(\frac{1}{n^{k-1/2}}\right)\ .$$

Similarly,
$$I_3 = O\left(\frac{1}{n^{k-\frac{1}{2}}}\right)\ .$$

In our final result we wish to get rid of the quantity δ. We first note that

$$\int_{-1+\delta}^{1-\delta} \psi(t)\,\frac{\sin(nx+\mu\pi)}{\sin x}\,dt \qquad [-\cos x = t]$$
$$= \int_{-1}^{1} \psi(t)\,\frac{\sin(nx+\mu\pi)}{\sin x}\,dt + O\left\{\int_{-1}^{-1+\delta}\frac{(t+1)^k}{\sqrt{1-t^2}}\,dt + \int_{1-\delta}^{1}\frac{(1-t)^k}{\sqrt{1-t^2}}\,dt\right\}$$
$$= \int_0^{\pi}\psi(-\cos x)\,\sin(nx+\mu\pi)\,dx + O\left(\frac{1}{n^{k+\frac{1}{2}}}\right)\ .$$

Further, writing
$$A_p^* = \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{(t-1)^p(t+1)^{p-1}}\,dt,$$
$$B_p^* = \frac{1}{\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{(t-1)^p(t+1)^{p}}\,dt,$$

we have
$$A_p'' = A_p^* + O\left\{\delta^{1+\min(k-p+\mu_0,\,k-p+1-\mu_0)}\right)$$
$$= A_p^* + O\left(\delta^{k-p+1+\mu_0}\right)$$

and

$$B_p'' = B_p^* + O(\delta^{k-p+1-|\mu_0|}).$$

Hence

$$a_p(n)A_p'' = a_p(n)A_p^* + O\left(\frac{1}{n^{2p-1-2|\mu_0|}} \cdot \frac{1}{n^{k-p+1+\mu_0}}\right)$$

$$= a_p(n)A_p^* + O\left(\frac{1}{n^{k+p-3|\mu_0|}}\right),$$

$$b_p(n)B_p'' = b_p(n)B_p^* + O\left(\frac{1}{n^{2p-1+2\mu_0}} \cdot \frac{1}{n^{k-p+1-|\mu_0|}}\right)$$

$$= b_p(n)B_p^* + O\left(\frac{1}{n^{k+p-3|\mu_0|}}\right).$$

So our final result is

$$(16) \qquad b_n'' = I_1 + I_2 + I_3$$

$$= \frac{2}{\pi}\int_0^\pi \psi(-\cos x)\sin(nx+\mu\pi)dx + O\left(\frac{1}{n^{k+\frac{1}{2}}}\right)$$

$$+ \sum_{p=1}^{k}\left\{a_p(n)A_p^* + b_p(n)B_p^* + O\left(\frac{1}{n^{k+p-3|\mu_0|}}\right)\right\}$$

$$+ O\left(\frac{1}{n^{2k-1}}\right) + O\left(\frac{1}{n^{k-\frac{1}{2}}}\right)$$

$$= \frac{2}{\pi}\int_0^\pi \psi(-\cos x)\sin(nx+\mu\pi)dx + \sum_{p=1}^{k}\left\{a_p(n)A_p^* + b_p(n)B_p^*\right\} + O\left(\frac{1}{n^{k-\frac{1}{2}}}\right).$$

From (14) and (16) our result follows.

We finally make some remarks concerning the formula (6). The first term in the right-hand member of (6) is not of much significance. In fact, let us write $\varphi(-\cos\ )-g_{2k}(-\cos x)=h(x)$ and let us suppose that $\varphi(t)$ is k times differentiable, not only at the points -1 and 1, but in the whole interval (-1,1). Then h(x) is k times differentiable in the interval $(0,\pi)$, whereas

$$h^{(p)}(-1) = h^{(p)}(1) = 0 \text{ for } p=0,1,2,\dots,k-1,$$

by the definition of $g_{2k}(t)$. Hence partial integration yields

$$\int_0^\pi h(x)\sin(nx+\mu\pi)dx = \frac{1}{n}\int_0^\pi h'(x)\cos(nx+\mu\pi)dx =$$

$$= \pm\frac{1}{n^k}\int_0^\pi h^{(k)}(x)\,\begin{matrix}\cos\\ \sin\end{matrix}(nx+\mu\pi)dx = O\left(\frac{1}{n^{k+\frac{1}{2}}}\right).$$

So, writing down the terms in the right-hand member of (6) up to the terms of order $O\left(\frac{1}{n^{k-\frac{1}{2}}}\right)$, we get the following result:

<u>If $\varphi(t)$ is k times differentiable in the whole interval [-1,1], then one has</u>

$$(17) \qquad b_n = \sum_{p=1}^{[(k+1)/2]} \left\{ a_p(n)A_p + b_p(n)B_p \right\} + O\left(\frac{1}{n^{k-\frac{1}{2}}}\right),$$

where $a_p(n), b_p(n)$, A_p, B_p are given by (7) - (10).

In the case $\mu=0$ (which we excluded above), however, only a term similar to the first term in the right-hand member of (6) appears. In fact, in this case b_n is simply a Fourier coefficient, viz.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

We shall now show how Lauwerier's result (3) can be deduced from our result (17). To this end it is necessary to suppose $k \geqq 2$, i.e. that $\varphi(t)$ is twice differentiable in the whole interval $[-1,1]$. Then, by (17),

$$b_n = a_1(n)A_1 + b_1(n)\, B_1 + O(n^{-3/2}).$$

We consider successively the quantities $A_1, B_1, a_1(n), b_1(n)$. Put $\varphi(-1)=\alpha$, $\varphi(1)=\beta$. Then

$$g_1(t) = \beta, \quad g_2(t) = \frac{\alpha}{2}(1-t) + \frac{\beta}{2}(1+t).$$

Hence, if A and B are given by (4) and (5),

$$A_1 = -A,$$
$$B_1 = \frac{1}{2\pi}\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \left\{ \frac{\varphi(t)}{t-1} - \frac{\varphi(t)}{t+1} + \frac{\alpha}{t+1} - \frac{\beta}{t-1} \right\} dt + \frac{\alpha-\beta}{2\sin\mu\pi}$$
$$= -\tfrac{1}{2}A - \tfrac{1}{2}B.$$

Next,
$$a_1(n) = \frac{1}{2\pi i}\int_{C_1+C_2} s^{-n-1}\left(\frac{1-s}{1+s}\right)^{2\mu} ds$$
$$= \frac{1}{2\pi i}\int_{C_1} s^{-n-1}\left(\frac{1-s}{1+s}\right)^{2\mu} ds + (-1)^n \frac{1}{2\pi i}\int_{C_2} s^{-n-1}\left(\frac{1-s}{1+s}\right)^{-2\mu} ds$$
$$= -\frac{\sin 2\mu\pi}{\pi}\left\{ \int_1^{\infty} s^{-n-1}\left(\frac{s-1}{s+1}\right)^{2\mu} ds + (-1)^{n+1}\int_1^{\infty} s^{-n-1}\left(\frac{s-1}{s+1}\right)^{-2\mu} ds \right\}.$$

The last integrals can be expanded as follows:
$$\int_1^{\infty} s^{-n-1}\left(\frac{s-1}{s+1}\right)^{2\mu} ds = \int_0^{\infty} e^{-nx}\left(\frac{e^x-1}{e^x+1}\right)^{2\mu} dx$$
$$= 2^{-2\mu}\int_0^{\infty} e^{-nx}x^{2\mu}dx(1+O(n^{-1})) = \frac{2^{-2\mu}}{n^{1+2\mu}}\,\Gamma(1+2\mu).(1+O(n^{-1})),$$

and similarly
$$\int_1^{\infty} s^{-n-1}\left(\frac{s-1}{s+1}\right)^{-2\mu} ds = \frac{2^{2\mu}}{n^{1-2\mu}}\Gamma(1-2\mu)(1+O(n^{-1})).$$

Hence

$$a_1(n) = \frac{2^{-2\mu}}{\Gamma(-2\mu)n^{1+2\mu}}(1+O(n^{-1})) + (-1)^n \frac{2^{2\mu}}{\Gamma(2\mu)n^{1-2\mu}}(1+O(n^{-1})).$$

Further

$$b_1(n) = -\tfrac{1}{2}\cdot\frac{1}{2\pi i}\int_{C_1} s^{-n-2}(1-s)^{2\mu}(s+1)^{2-2\mu}ds + $$
$$+ (-1)^n \tfrac{1}{2}\frac{1}{2\pi i}\int_{C_1} s^{-n-2}(1+s)^{2\mu}(1-s)^{2-2\mu}ds$$
$$= + \frac{\sin 2\mu\pi}{2\pi}\int_1^\infty s^{-n-2}(s-1)^{2\mu}(s+1)^{2-2\mu}ds \quad (1+O(n^{-1}))$$
$$= - \frac{2^{1-2\mu}}{\Gamma(-2\mu)n^{1+2\mu}}(1+O(n^{-1})).$$

So, finally,

$$b_n = -A\left\{\frac{2^{-2\mu}}{\Gamma(-2\mu)n^{1+2\mu}} + (-1)^n\frac{2^{2\mu}}{\Gamma(2\mu)n^{1-2\mu}}\right\} + (A+B)\frac{2^{-2\mu}}{\Gamma(-2\mu)n^{1+2\mu}} + $$
$$+ O(n^{-2+2|\mu_0|})$$
$$= (-1)^{n+1}\frac{2^{2\mu}A}{\Gamma(2\mu)n^{1-2\mu}} + \frac{2^{-2\mu}B}{\Gamma(-2\mu)n^{1+2\mu}} + O(n^{-2+2|\mu_0|}).$$

This proves (3).

Finally we wish to show how one can expand the sum in the right-hand member of (17) to ascending powers of $1/n$. For this purpose we define

$$c(m,\gamma,\delta) = \int_1^\infty \frac{1}{s^{m+1}}(s-1)^\gamma (s+1)^\delta ds,$$

where m is a positive integer and γ and δ are complex numbers with $\mathrm{Re}\,\gamma > -1$. Then

$$a_p(n) = \frac{1}{2\pi i}\int_{C_1}\frac{1}{s^{n+1}}\left(\frac{1-s}{1+s}\right)^{2\mu}\frac{(s+1)^{2p-2}(s-1)^{2p-2}}{(2s)^{2p-2}}ds + $$
$$= (-1)^n\frac{1}{2\pi i}\int_{C_1}\frac{1}{s^{n+1}}\left(\frac{1-s}{1+s}\right)^{-2\mu}\frac{(s+1)^{2p-2}(s-1)^{2p-2}}{(2s)^{2p-2}}ds$$
$$= \frac{\sin 2\mu\pi}{\pi 2^{2p-2}}\left\{-c(n+2p-2,2\mu+2p-2,-2\mu+2p-2)+(-1)^n c(n+2p-2,-2\mu+2p-2,2\mu+2p-2)\right\},$$

$$b_p(n) = -\frac{1}{2\pi i}\int_{C_1}\frac{1}{s^{n+1}}\left(\frac{1-s}{1+s}\right)^{2\mu}\frac{(s+1)^{2p}(s-1)^{2p-2}}{(2s)^{2p-1}}ds$$
$$-(-1)^{n+1}\frac{1}{2\pi i}\int_{C_1}\frac{1}{s^{n+1}}\left(\frac{1-s}{1+s}\right)^{-2\mu}\frac{(s-1)^{2p}(s+1)^{2p-2}}{(2s)^{2p-1}}ds$$
$$= \frac{\sin 2\mu\pi}{\pi 2^{2p-1}}\left\{c(n+2p-1,2\mu+2p-2,-2\mu+2p)+(-1)^n c(n+2p-1,-2\mu+2p,2\mu+2p-2)\right\}.$$

Hence

$$(18)\quad b_n = \frac{\sin 2\mu\pi}{\pi}\sum_{p=1}^{[(k+1)/2]}\Big[\Big\{-2^{2-2p}A_p c(n+2p-2,2\mu+2p-2,-2\mu+2p-2)$$

$$+2^{1-2p}B_p\, c(n+2p-1,2\mu+2p-2,-2\mu+2p)\Big\}$$

$$+(-1)^n\Big\{2^{2-2p}A_p\, c(n+2p-2,-2\mu+2p-2,2\mu+2p-2)+2^{1-2p}B_p c(n+2p-1,-2\mu+2p,2\mu+2p-2)\Big\}$$

$$+O(n^{-k+\frac{1}{2}}).$$

Here the asymptotic behaviour of the c can easily be determined. One has

$$c(m,\gamma,\delta) = \int_1^\infty s^{-m-1}(s-1)^\gamma\ (s+1)^\delta\, ds$$

$$= 2^\delta\int_0^\infty e^{-mx}\ (e^x-1)^\gamma\ \Big\{1+\frac{e^x-1}{2}\Big\}^\delta\ dx$$

$$= 2^\delta\ m^{-\gamma-1}\Gamma(\gamma+1) + d_1(\gamma,\delta)m^{-\gamma-2}\Gamma(\gamma+2) + \ldots$$

$$+ d_{k-2}(\gamma,\delta)m^{-\gamma-k+1}\Gamma(\gamma+k-1) + O(m^{-\gamma-k}),$$

where $d_1(\gamma,\delta)$, $d_2(\gamma,\delta)$,... are constants not depending on m.